

Flag vectors

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Abstract

This paper defines for each object X that can be constructed out of a finite number of vertices and cells a vector fX lying in a finite dimensional vector space. This is the flag vector of X . It is hoped that the quantum topological invariants of a manifold M can be expressed as linear functions of the flag vector of the i -graph that arises from any suitable triangulation T of M . Flag vectors are also defined for finite groups and more generally for n -ary relations. Some problems, and suggested connections with other constructions, particularly that of the associahedron and so on, conclude the presentation.

1 Introduction

This paper defines for each object X that can be constructed out of vertices and cells a vector fX lying in a finite dimensional vector space. This is the flag vector of X . An example that indicates the importance of this problem follows.

Suppose that M is a compact topological manifold, say of dimension n . Now let T be a triangulation of M . Because T determines M , any topological invariant of M can be calculated via T , at least in principle. Considered abstractly, T can be described as follows. First, let V be the set of points of M that are vertices of cells (in fact simplices) in the triangulation. Each n -dimensional ‘triangle’ or simplex in T determines an $(n + 1)$ -element subset of V . Let C be the collection of all such subsets of V . Assume also that each cell c in C corresponds to just one simplex of T . It now follows that the combinatorial object $G = (V, C)$, which is an example of what is known as an $(n + 1)$ -graph, determines T and thus M up to equivalence. This paper will define a flag vector fG for all $(n + 1)$ -graphs G .

Now suppose that $v = v(M)$ is a numeric (or vector) valued topological invariant of n -manifolds. For each $(n + 1)$ -graph G a topological realisation $|G|$ can be produced, and if $|G|$ is a manifold M then we will define $v(G)$ to be $v(M)$. The next three definitions describe the relationship we seek between the topological invariant v and the flag vector f .

Definition 1 *Suppose that $v(G_1) = v(G_2)$ whenever $fG_1 = fG_2$, for any G_i for which $|G_i|$ is a manifold. In that case we will say that $v(G)$ is a function of fG .*

Definition 2 *Suppose that v is a function of fG and in addition that whenever a linear relation such as*

$$\lambda_1 fG_1 + \dots + \lambda_r fG_r = 0 \quad \lambda_i \in \mathbf{R}$$

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holds between flag vectors of graphs for which $|G_i|$ is a manifold M_i then the corresponding relation

$$\lambda_1 v(M_1) + \dots + \lambda_r v(M_r) = 0$$

holds between the values of v , then we will say that $v(G)$ is a linear function of fG .

Definition 3 Suppose that fG is given as a point in a vector space F , and that v is in the above sense a linear function of f . Any linear function on F that agrees with v when $|G|$ is a manifold will be called a formula for v in terms of f . (Such a formula will not be unique, unless F is spanned by the fG for which $|G|$ is a manifold.)

The methods of quantum topology provide a interesting and steadily growing collection of topological invariants. This is an extremely active area of research. These invariants are usually discovered family by family, and in more or less explicit form. The difficult task is usually to demonstrate topological invariance. The flag vector approach to topological invariants is wholesale rather than retail, or top-down rather than bottom-up. It is in part inspired by the Vassiliev theory of knot invariants, which is similarly a wholesale approach.

Each definition of a flag vector defines a family of topological invariants, namely those that are linear functions of the flag vector. Determining such functions is of course not likely to be so easy. Loosely speaking, the flag vector is a haystack in which one hopes to find interesting needles. For such to be useful, it should be neither too large nor too small. One wishes to narrow down the search, without discarding any needles.

The definitions to be given in this paper will apply not only to $(n+1)$ -graphs but more generally to any object that is built up out of a finite number of vertices and cells, or can be so represented. Thus, a flag vector will be defined for finite groups, where the vertices are the elements and the cells are the equations $ab = c$ that hold between the elements.

This paper is organised as follows. The next two sections define first the shelling vector and then the flag vector of an i -graph, and the following section describes by means of examples the changes that must be made to accomodate more general vertex-and-cell objects. Finally, there is a summary and the statement of some open problems.

The preprints [2, 3] are perhaps best thought of as preliminary forms of this paper. The first does, however, contain additional material. This paper can usefully be read in conjunction with [5], which deals with ordinary or 2-graphs.

2 The shelling vector

Throughout this section and the next G will be an i -graph, or in other words a possibly empty collection C of i -element subsets of a vertex set V . Traditionally, such are called edges.

Definition 4 A shelling σ of G is simply an ordering v_1, \dots, v_r of the vertex set V . The j -th link L_j of the shelling σ is the $(i-1)$ -graph consisting of all $(i-1)$ -element subsets c of v_{j+1}, \dots, v_r such that $c \cup \{v_j\}$ is a cell of G .

The definition of the shelling vector is recursive. Each shelling σ has links L_i which, by assumption, will already have a shelling vector. We ‘multiply’ these together, and take the sum over all shellings.

Definition 5 Let G be an i -graph, with $i > 0$. The shelling vector $\tilde{f}G$ is the sum

$$\sum_{\sigma} \tilde{f}L_1 \otimes \dots \otimes \tilde{f}L_r$$

over all shellings of the tensor product of the shelling vectors of the links.

Definition 6 There is only one set with zero elements, namely the empty set. Thus, on any vertex set V there are only two 0-graphs. One has no cells, and the other has the empty set as its only cell. The shelling vectors of these graphs will be defined to be the symbols a and b respectively.

We have now defined a shelling vector for i -graphs. It can be thought of as a formal sum of words in a and b . (The length of these words will depend on both the size r of the vertex set and the size i of the subsets being used. For example, for 1-graphs the length is r , while for 2-graphs the length is $r(r+1)/2$. In general, the length for i -graphs is the sum of the lengths for the $(i-1)$ -graphs that are the links.)

Almost certainly the shelling vector is too large for our purposes. If the following is true then any topological invariant whatsoever of M will be a linear function of the shelling vector.

Problem 7 Suppose G_1, \dots, G_r is any collection of i -graphs, no two of which are equivalent. Are the shelling vectors $\tilde{f}G_1, \dots, \tilde{f}G_r$ linearly independent?

3 The flag vector

The flag vector will again be a recursive sum over all shellings, but this time of products of link contributions. In the shelling vector, each link contributed its own shelling vector. For the flag vector, the contribution made by a link L_i will be not the whole of the flag vector $\tilde{f}L_i$ of the link, but some reduction $\overline{f}L_i$ of this quantity.

Definition 8 Let G be an i -graph, with $i > 0$. The flag vector $\tilde{f}G$ is the sum

$$\sum_{\sigma} \overline{f}L_1 \otimes \dots \otimes \overline{f}L_r$$

over all shellings, where the link contribution $\overline{f}L_i$ will be defined later. The flag vector of a 0-graph is either a or b , as with the shelling vector.

We will first give the definition of the link contribution, and then we will motivate it. Let $F = F_i = F_{i,r}$ be the vector space in which $\tilde{f}G$ naturally lies. The link contribution $\overline{f}L_j$ will be the residue $\overline{f}L_j$ of $\tilde{f}L_j$ in a certain quotient \overline{F} of F .

Definition 9 Suppose G is an i -graph. Two distinct cells c_1 and c_2 of G are disjoint if they do not have a vertex in common. Suppose that two such cells have been chosen. Let G_{+-} and G_{-+} denote the result of removing c_2 and c_1 respectively from G . Let G_{--} denote the result of removing both c_1 and c_2 , and set G_{++} to be G itself. Recall that $\tilde{f}G$ will lie in a vector space F . Define the link space \overline{F} to be the quotient of F by the subspace spanned by

$$\tilde{f}G_{++} - \tilde{f}G_{+-} - \tilde{f}G_{-+} + \tilde{f}G_{--}$$

for all G , and all pairs (c_1, c_2) of disjoint cells in G . Now define the link contribution $\overline{f}G$ of G to be the residue of $\tilde{f}G$ in \overline{F} .

For 0-graphs the flag vector is by definition the same as the shelling vector, namely either a or b . For 1-graphs the links are 0-graphs, and as such graphs do not have two cells to be disjoint, the link contributions are again either a or b . There are up to equivalence $(r + 1)$ distinct 1-graphs on r vertices, and it is easily seen that their flag vectors are linearly independent.

For ordinary or 2-graphs the disjoint cells rule comes in, to reduce the link contribution from $fL = \tilde{f}L$ to $\bar{f}L$. This is how it goes. Just for this paragraph, let $[n]$ denote the link contribution due to an n -celled 1-graph on some fixed number r of vertices. The equation

$$[n + 2] - [n + 1] - [n + 1] + [n] = 0$$

follows from the disjoint cell rule, and all relations arise in this way. If we write the equation as

$$[n + 2] - [n + 1] = [n + 1] - [n]$$

then its meaning becomes clearer. The change made by adding a cell to the link does not depend on the number of cells in the link, at least in the present situation. It is easily seen that the vectors

$$a = [0], \quad b = [1] - [0]$$

provide a basis for the link space, and that $[n]$ is equal to $a + nb$. (The symbols a and b are not the same as those used in shelling vectors.)

We can now motivate the disjoint cell rule for the link contribution. The previous paragraph shows that for 2-graphs the definition in this paper agrees with that in [5]. The results in [5] indicate that the correct definition has been found, at least for 2-graphs, and gives some insight into how the disjoint cell works in this case.

When the link is a 1-graph, distinct cells are automatically disjoint. Elsewhere [4] the concept of independent regional change has proved to be useful. The disjoint cell rule is simply another application of this principle.

4 Further examples

We have defined, for each i -graph, a flag vector fG . The same process can be applied to other combinatorial objects, built up out of vertices and cells. For example, to study the topology of oriented manifolds, one will need to study *oriented* i -graphs. (Such is an i -graph, where each cell has been given an orientation. An orientation is an ordering of the vertices, up to an even permutation.)

Once suitable sign conventions have been established, the result of removing a vertex from an oriented i -cell will be an oriented $(i - 1)$ -cell. This works for $i \geq 3$. For $i = 2$ there will be only a single ordering for the resulting 1-cell, and so some other convention must be used instead. Better however is to change the definition of an orientation. Instead, say that an orientation of a cell is a rule that assigns a sign to each ordering of the vertices, in a manner that respects even and odd permutations of the vertices. When this is done, both 1-cells and 0-cells can be oriented. (In both cases, there is only one ordering of the vertices available.)

In the unoriented case, the inductive definition of the shelling and flag vectors was founded on the values a and b for 0-graphs. In the oriented case, there are two possible sorts of 0-cell, which can conveniently be denoted by b_+ and b_- . These, together with a , will found the inductive definition for oriented i -graphs.

Something similar can be done for manifolds with a boundary. Define an *i -cell with boundary* to be an ordinary i -cell, together with a possibly empty subset of the vertex set, which is the *boundary*

of the cell. Define an i -graph with boundary to be a set of i -cells with boundary, where as ordinary cells the i -cells are distinct. Clearly, the result of removing a vertex from an i -cell with boundary will be an $(i - 1)$ -cell with boundary.

However, there is more. We will wish to record whether or not the removed vertex was a boundary vertex. (In the oriented case, we used the removed vertex to choose an orientation for the resulting cell.) We can record this as a label attached to the $(i - 1)$ -cell that results from removing a vertex from an i -cell with boundary. Successive removal of vertices will result in a word being written on the label, and so the induction will be founded on the 0-cells a and b_w , where a is the ‘empty’ 0-graph, and where b_w is the ‘full’ 0-graph whose label w is a word of length i in say 0 and 1, which records the removal of boundary vertices.

A flag vector for finite groups was promised in the introduction. The vertices will be the elements of G . We will think of the group law as the ternary relations $R = R(x, y, z)$ whose triples (a, b, c) solve the equation $xy = z$ that represents the group law.

Loosely speaking, the cells will be the triples (a, b, c) that belong to R . Given a shelling of the vertices, we will obtain the i -th link by first restricting the relation to the vertices not already removed, and then taking only those cells that use the i -th vertex v_i . However, because an unordered triple $\{a, b, c\}$ may support several ordered triples that satisfy R , and because a doubleton $\{a, b\}$, or even a singleton $\{a\}$ may support an ordered triple satisfying R , a more careful approach is needed.

We will instead follow the logic used in §2 to define the shelling vector, but apply it instead to an arbitrary ternary relation R . When a vertex v is removed from an i -graph G , the link was defined to be the $(i - 1)$ -graph whose cells c become cells of G when v is added to c . A cell of an i -graph is an unordered set of i elements, and similarly for $(i - 1)$ graphs. Adding a vertex to c is then just a matter of adding an element to an unordered set. This is straightforward.

We now apply the same logic to the ternary relation R . Here order and location are important. Suppose that (a, b, c) is a triple that satisfies R , with a , b and c distinct. The result of removing $v_i = a$ can conveniently be denote by $(\mathbf{1}, b, c)$, where the $\mathbf{1}$ is a *placeholder* that indicates that a vertex was removed at the first step. Similarly, removing b from (a, b, b) will produce $(a, \mathbf{1}, \mathbf{1})$. The following two definitions are easily seen to be equivalent.

Definition 10 *Suppose that R is a ternary relation on a vertex set V , and that a is a vertex of R . Now replace a throughout by the placeholder $\mathbf{1}$ in both R and V , and discard from R the triples that do not contain the placeholder $\mathbf{1}$. This new ternary relation R_a is defined to be the link of R at a . (The i -th link of a shelling is much as before R'_a , where R' is R restricted to the vertices that remain before $v_i = a$ is removed.)*

Definition 11 *The link R_a consists of the ordered triples on $V \setminus \{a\} \cup \{\mathbf{1}\}$ that contain at least one placeholder, and which belong to R when $\mathbf{1}$ is replaced by a .*

Although the link R_a is still a ternary relation, it is rather different from its parent R in that its relations all contain the placeholder $\mathbf{1}$, which is not a vertex of R . This must be taken into account when a second vertex b is removed, to compute the flag vector of the link.

Definition 12 *Suppose that a and b are distinct. The second-order link R_{ab} is obtained from a first order link R_a in the following way. In R_a replace b throughout by the placeholder $\mathbf{2}$, and discard from R_a all triples that do not contain at least two placeholders. The result is R_{ab} . Similarly, the third order link R_{abc} is obtained from R_{ab} by replacing c by the placeholder $\mathbf{3}$ and discarding from R_{ab} the triples that do not contain at least three placeholders.*

As with R_a and R , so R_{ab} consists of the triples in $V \setminus \{a, b\} \cup \{\mathbf{1}, \mathbf{2}\}$ with at least two placeholders, that belong to R_a when $\mathbf{2}$ is replaced by b . Much the same holds for R_{abc} and R_{ab} . (This use of placeholders will define links not only for ternary relations, but for n -ary relations for any n , and for more general objects yet.)

When we get down to the third order link R_{abc} , the vertex set V will have disappeared completely. All that remains will be a ternary relation R_{abc} on the placeholders $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$. Not all such will arise in this way. For example $(\mathbf{1}, \mathbf{3}, \mathbf{3})$ is forbidden because on replacing $\mathbf{3}$ by c the result $(\mathbf{1}, c, c)$ does not have at least two placeholders. The inductive definition of the shelling vector for ternary relations is now complete. It is founded on the ternary relations on $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ that are not as just described forbidden.

To go on to define the flag vector we must know what a cell is, so that we can talk of disjoint pairs of cells. But first we consider a detail that arises for all flag vectors, except those of i -graphs.

Definition 13 *Whatever a cell may be, its support consists of the set all vertices of V (but not placeholders) that appear in the cell.*

For i -graphs, a cell is determined by its support. In the oriented case, and in other situations, there can be several different cells with the same support set. Hitherto we have been defining G_{+-} and so on via the appearance or non-appearance of cells in a disjoint pair. For i -graphs this was the only choice.

However, for other objects we might wish not to remove cells, but merely to change them, without altering the support. It is easily seen that this additional freedom produces no new relations, in the construction of the link space \overline{F} from the flag vector space F via the disjoint pair of cells rule. This is because to change a cell is to first remove it, and to then replace it by the new value.

Definition 14 *Let R' be a link for a relation R . For example, R' might be R_a or R_{ab} . A simple change consists of the addition to or removal from R' of an n -tuple. Its support is the support of the cell.*

The next definition is slightly subtle, because for relations the support of a cell in the link might be empty. The triples $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ are examples of this. Its effect is to group together such cells in the link into a single compound cell, similar to the placeholder relations that are used to found the induction.

Definition 15 *Let R' be as before, and suppose that two simple changes are given, whose support sets are distinct and disjoint. As before, this produces four variants R'_{++} , R'_{+-} , R'_{-+} and R'_{--} of $R' = R'_{++}$. The disjoint cell rule defines the link space \overline{F} to be the quotient of the vector space F that fR' naturally lies in by the subspace spanned by*

$$fR'_{++} - fR'_{+-} - fR'_{-+} + fR'_{--}$$

for all such R' equipped with such a pair of simple changes. As before, the link contribution $\overline{f}R'$ of R' is defined to be the residue of fR' in \overline{F} .

This completes the definition of the flag vector for n -ary relations, and group laws $xy = z$ in particular. It should now be clear how to define a flag vector for anything that is built out of vertices and cells.

5 Summary and conclusions

In this paper we have assigned a flag vector fG to i -graphs, groups and other objects constructed out of vertices and cells, such as n -ary relations. The example of quantum topology shows the importance of being able to make such a construction. In this final section we discuss some questions whose solution will have some bearing on the fruitfulness of these definitions. We begin with quantum topology and the shelling concept.

Problem 16 *Can existing quantum topological invariants be expressed as linear functions of the flag vector?*

Some quantum topology invariants can be computed with the aid of a generic height function h (a Morse function) on the manifold being studied. Such will have a finite number of critical points, each locally equivalent to a non-degenerate quadratic form. Now let T be a triangulation of M that is compatible with the height function h . More exactly, adjust h by composing with a monotonic function so that the vertices have as heights the integers 1 through to the number r of vertices, and insist that h is topologically equivalent to its linear interpolation onto the cells of the triangulation.

In other words, at least some of the time a shelling σ of a hypergraph G can represent a Morse function h on a manifold M . This establishes another point of contact between the two theories, and gives some new insight into the significance of the concept of a shelling. Incidentally, in convex polytope theory a shelling of a simplicial polytope is equivalent (under polarization) to a generic height function on a simple polytope which in turn, via the moment map, can induce a Morse function on a projective toric variety. (The induced function is Morse if the toric variety is a nonsingular.)

Assume that G and σ represent a Morse function h on a manifold M . In general terms, certain quantum invariants v of M can be expressed as very special linear combinations of numbers λ_i that can be computed from G and σ . On the other hand, each shelling σ of G makes a contribution $f_\sigma G$ to fG . If the λ_i turn out to be linear functions of $f_\sigma G$, then this is evidence for $v(M)$ being a linear function of fG . For example, if every shelling σ represents a Morse function on M , then the result would follow.

Next we ask to what degree the flag vector distinguishes inequivalent objects. The following successively stronger questions are an example of what we might wish for.

Problem 17 *Suppose $fG_1 = fG_2$. Does it follow that G_1 and G_2 are equivalent?*

Problem 18 *Let Δ be the convex hull of fG , as G runs over a class of objects. Are the vectors fG vertices of Δ , and are they distinct for distinct G ?*

Problem 19 *Is there a natural inner product on the space F in which vectors fG (and Δ) lies, such that the fG are distinct and lie on a sphere?*

If the last problem has a positive solution, it is perhaps the easiest way to resolve the first two. Although such a result might be thought unlikely, there are already examples of combinatorially derived polytopes, whose vertices lie on a sphere. The permutahedron, the associahedron and the permuto-associahedron are examples of this. Initially [6], the permuto-associahedron was a combinatorially labelled cell complex whose realization, after some work, was found to be a sphere.

(This result was useful in algebraic topology.) Later, it was found that this cell complex could be realized as the boundary of a convex polytope whose vertices, for a suitably cunning construction, would lie on a sphere. This phenomenon is at present unexplained, and is rather strange. It involves fiber polytopes [1]. Lecture 9 of [7] is a good first reference. The construction is there presented as the problem of constructing a polytope with prescribed and rather special combinatorics.

This problem can be explored in several ways. One is to study the flag vectors of rather special and simple objects, such as binary relations or partial orders on a small number of vertices. (A graph is, of course, a special type of relation among its vertices.) If the result is a convex polytope, whose edges and so forth have combinatorial significance, then we are encouraged. Another is to define a flag vector for the combinatorial objects that are the vertices of the associahedron and so forth. This will require some thought, for such objects are not at least on the face of it n -ary relations or the like. If all is well, this should give an alternative approach to the presently unexplained construction of the permuto-associahedron.

This paper consists largely of definitions. Its purpose is to delimit an area of study, rather than to obtain results in that area. The flag vector at present stands somewhat apart from the rest of mathematics, and other than [5] results are not yet available. A number of worked examples, and investigation of some of the simpler problems, seems to be the next step.

Finally, there is another approach. An object consisting of vertices and cells can be shelled by removing the vertices, and thus removing the cells. This is the path we have followed. The other approach, described for graphs at the end of [5], is to shell the object by removing the *cells* one at a time. This approach is at the time of writing completely unexplored.

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